UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

PHYSICS	**AUCKLAND PARK KINGSWAY CAMPUS**

PHY002B

SUPPLEMENTARY EXAMINATION
01 December 2014
08:30-11:30

Static and Dynamic Electromagnetism

EXAMINER: **Prof S Razzaque**

MODERATOR: Dr CJ Sheppard

TIME: 3 Hours **MARKS: 150**

Please read the following instructions carefully:

1. Answer all the questions
2. No programmable calculator allowed
3. This paper consists of 6 pages (including this page)

Some useful constants and formulae

$c = 2.998 \cdot 10^{8}$ m/s

$e = 1.602 \cdot 10^{-19}$ C

$\varepsilon_0 = 8.854 \cdot 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$

$\mu_0 = 1.257 \cdot 10^{-6}$ kg m/C^2

$$\int_S \mathbf{F} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{F})\, dv$$

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

$$\phi_2 - \phi_1 = \int_C \nabla\phi \cdot d\mathbf{s}$$

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

$$\phi = -\int \mathbf{E} \cdot d\mathbf{s}$$

$$\mathbf{E} = -\nabla\phi$$

$$U = \frac{\epsilon_0}{2} \int E^2\, dv$$

$$Q = C\phi$$

$$U = \frac{1}{2} C\phi^2$$

$$J = Nev$$

$$I = \int \mathbf{J} \cdot d\mathbf{a}$$

$$\operatorname{div} \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

$$R = \frac{\rho L}{A}$$

$$P = IV = I^2 R$$

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

$$\mathcal{E} = \frac{1}{q} \int \mathbf{f} \cdot d\mathbf{s}$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathcal{E}_{21} = -M_{21} \frac{dI_1}{dt}$$

$$\mathcal{E}_{11} = -L_1 \frac{dI_1}{dt}$$

$$U = \frac{1}{2} L I^2$$

$$U = \frac{1}{2\mu_0} \int B^2\, dv$$

$$\tilde{I} = Y\tilde{V}$$

$$\tilde{V} = Z\tilde{I}$$

$$\tilde{Z}_C = (i\omega C)^{-1},\ \tilde{Z}_L = i\omega L$$

$$\overline{P}_R = \frac{V_{\mathrm{rms}}^2}{R}$$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

$$E_0 = \frac{B_0}{\sqrt{\mu_0 \epsilon_0}} = cB_0$$

$$S = \epsilon_0 \overline{E^2} c$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z},$$

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$$

Question 1 Marks: 7+8+10 = 25

a. What is electric flux? Define in words and mathematically. Calculate the flux through a sphere of radius R due to a point charge Q placed at the centre of the sphere.

b. The potential difference between two points in an electric field **E** is given by the line integral $\phi \equiv \phi_{21} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s}$, where $d\mathbf{s}$ is the infinitesimal displacement vector. Using analogy with the gradient of a scalar function, show that $\mathbf{E} = -\nabla\phi$. You may assume Cartesian coordinates with $\nabla \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

c. A uniformly charged circular disk of radius a has surface charge density σ. You may assume the disk is lying on the xz-plane. Draw an appropriate diagram and calculate
   i. the potential at any point on the axis of symmetry of the disk,
   ii. the potential at the centre of the disk, and
   iii. the total energy associated with the electric field.

Question 2 Marks: 10+8+7=25

a. The capacitance C is defined as the total charge Q of an object divided by the potential ϕ, that is $C = Q/\phi$. Consider a uniformly charged conducting sphere of radius a, with a total charge Q.
   i. What is the potential at the surface of the sphere?
   ii. What is the capacitance of the sphere?
   iii. N spherically symmetric charged raindrops with radius a all have the same potential. Assume that they are far enough apart so that the charge distribution on each is not affected by the others. What is the total capacitance of this system? How does this capacitance compare with the capacitance in the case where the drops are combined into one big drop?

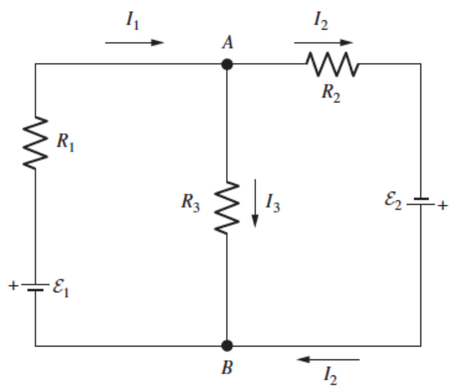
b. A vacuum capacitor consists of two coaxial cylinders of outer radius a, inner radius b and length L.
   i. Find the capacitance assuming $L >> a - b$ so that the end corrections may be neglected.
   ii. With a given radius a for the outer cylindrical shell, that will be able to store the greatest amount of electrical energy per unit length, subject to the constraint that the electric field strength at the surface of the inner cylinder may not exceed E_0. What

radius b should be chosen for the inner cylindrical conductor, and how much energy can be stored per unit length?

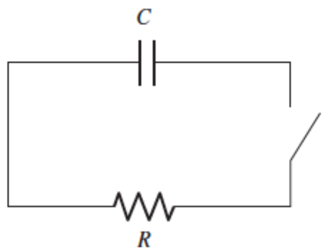
c. A 100 pF capacitor is charged to 100 volts. After the charging battery is disconnected, the capacitor is connected in parallel with another capacitor. If the final voltage is 30 volts, what is the capacitance of the second capacitor? How much energy was lost, and what happened to it?

Question 3 Marks: 9+7+9=25

a. What is current density (**J**)? How is it related to the electric current (I)? Show that $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ for a total charge $\int_V \rho \, dV$ inside a volume V, with charge density ρ, at any instant.



b. The circuit shown in figure contains two batteries with electromotive force $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. In each of the conventional battery symbols shown, the longer line indicates the positive terminal. Assume that R_1 includes the internal resistance of one battery, R_2 that of the other. Supposing the resistances are known, what are the currents in this network?



c. An RC circuit is shown in the figure. The capacitor has been charged initially and is now discharging through the resitor. What are the charge Q on the capacitor and the current I in the circuit, as functions of time?

Question 4 Marks: 6+11+8=25

a. Draw a clear, labeled diagram showing a straight conducting wire along the x-axis. A current I is flowing towards the $-x$ direction in the wire.
   i. Show the electron velocity direction in the wire.
   ii. What is the magnetic field direction and magnitude at a radial distance r from the wire?
   iii. What is the Lorentz force due to this magnetic field?

b. The magnetic field **B** can be expressed in terms of a vector potential **A** as $\mathbf{B} = \nabla \times \mathbf{A}$.

i. By comparing with the Poisson's equation of electrostatics $\nabla^2\phi(x,y,z) = -\rho(x,y,z)/\varepsilon_0$, where ϕ is the scalar potential and ρ is the volume charge density, show that the vector potential can be expressed in terms of current density **J** as

$\mathbf{A}(x_1,y_1,z_1) = \dfrac{\mu_0}{4\pi}\int_{V_2}\dfrac{\mathbf{J}(x_2,y_2,z_2)}{r_{21}}dv_2$, where V_2 is the volume with current density.

ii. What is the vector potential of a long straight wire carrying a current in the positive z direction? [Hint: use cylindrical coordinates.]

c. A spherical shell with radius R and uniform surface charge density σ spins with angular frequency ω around a diameter. Find the magnetic field at the center using the Biot-Savart law.

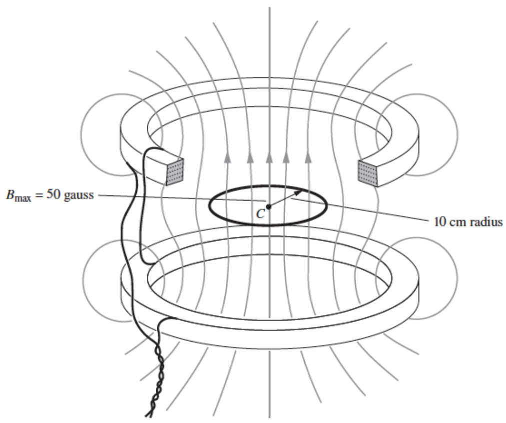
Question 5 Marks: 14+11=25

a. The magnetic flux through a closed surface S is defined as $\Phi(t) = \int_S \mathbf{B}\cdot d\mathbf{a}$ where **B** is the magnetic field and d**a** is the surface vector element. The time derivative of this flux is negative of the electromotive force (emf).

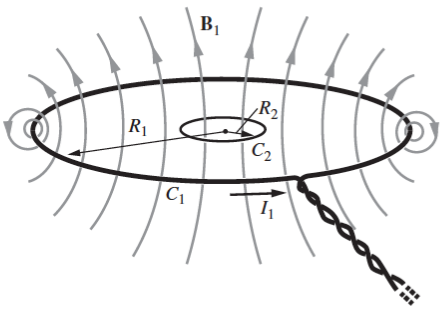
i. Show that Faraday's law relating electric and magnetic field follows from the electromotive force.

ii. An infinite solenoid has radius R and n turns per unit length. The current grows linearly with time, according to $I(t) = Ct$. Use the integral form of Faraday's law to find the electric field at radius r, both inside and outside the solenoid. Then verify that your answers satisfy the differential form of Faraday's law.

iii. The magnetic field between the coils shown in the figure is B_{max} sin($2\pi\omega t$) due to alternate current with cycle ω = 60/s and B_{max} = 0.005 T. Calculate the induced emf in the loop of radius 10 cm placed in between the coils at any time t.

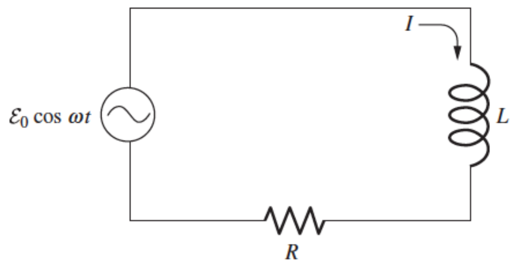


b. What is mutual inductance? What is its unit and how it is defined? Two coplanar, concentric rings: a small ring C_2 and a much larger ring C_1 are shown in the figure. Assuming $R_2 \ll R_1$, what is the mutual inductance M_{21}?



Question 6 Marks: 9+7+9=25

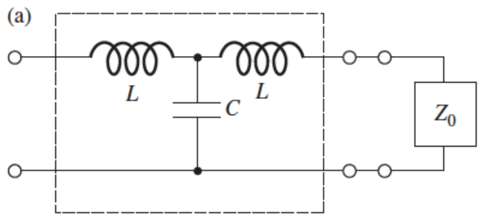
a. An *RL* circuit with alternating electromotive force $\varepsilon = \varepsilon_0 \cos \omega t$ is shown in the figure, where ω is the angular frequency. The current through the circuit at any time t can be described as $I(t) = I_0 \cos(\omega t+\phi)$, where ϕ is a phase shift.



i. Derive the initial current I_0 and phase ϕ.

ii. Draw a graph of ε and I versus time.

b. The box shown in the figure with four terminals contains a capacitor C and two inductors of equal inductance L connected as shown. An impedance Z_0 is to be connected to the terminals on the right. For given frequency ω, find the value that Z_0 must have if the resulting impedance between the terminals on the left (the "input" impedance) is to be equal to Z_0.



c. Consider the two oppositely traveling electric-field waves $\mathbf{E}_1 = \hat{\mathbf{x}} E_0 \cos(kz - \omega t)$ and $\mathbf{E}_1 = \hat{\mathbf{x}} E_0 \cos(kz - \omega t)$. The sum of these two waves is another standing wave $2\hat{\mathbf{x}} E_0 \cos kz \cos \omega t$.

i. Find the magnetic field associated with this standing electric wave by finding the **B** fields associated with each of the above traveling **E** fields, and then adding them.

ii. Find the magnetic field by instead using Maxwell's equations to find the **B** field associated with the standing electric wave, $2\hat{\mathbf{x}} E_0 \cos kz \cos \omega t$.